

SKEW CHROMATIC INDEX OF 2-ROOTED SIBLING TREES AND CYCLIC SNAKE GRAPHS

Joice Punitha M. and S. Rajakumari*

Department of Mathematics,
Bharathi Women's College(Auto.), Chennai-600108, INDIA
E-mail: joicepunitha@gmail.com

*Department of Mathematics,
R.M.D. Engineering College, Kavaraipettai-601206, INDIA
E-mail: srajakumari23@yahoo.com

(Received: May 8, 2018)

Abstract: A skew edge coloring of a graph G is defined as a set of two edge colorings such that no two edges are assigned the same unordered pair of colors. The skew chromatic index $s(G)$ is the minimum number of colors required for a skew edge coloring of G . In this article, we develop an algorithm for skew edge coloring of 2-rooted sibling trees and cyclic snake graphs. The minimum number of colors k which is known as the skew chromatic index is determined depending upon the number of edges of G . Furthermore, it is proved that the bound on the skew chromatic index $s(G) \geq \max\{\Delta(G), k(|E(G)|)\}$ is sharp for the family of graphs considered for skew edge coloring.

Keywords and Phrases: Skew edge coloring, skew chromatic index, 2-rooted sibling tree, cyclic snake graphs, NP -complete.

2010 Mathematics Subject Classification: 05C15.

1. Introduction

Edge coloring problem is one of the fundamental problems on graphs, which appears in various scheduling problems like file transfer problems on computer networks. It has wide variety of applications like job scheduling, timetable designing, and assignment of frequencies in wireless communications etc. They are well studied in both computer science and mathematics [9]. Let $G = (V, E)$ be a finite, simple, connected undirected graph with vertex set V and edge set E . A proper edge coloring of a graph G is an assignment of colors to the edges of G so that no

two adjacent edges are assigned the same color [3]. The minimum number of colors required for an edge coloring of G is the edge chromatic number or the chromatic index and is denoted by $\chi'(G)$. Let $\Delta(G)$ denote the maximum degree of vertices of a graph G . It has been proved by Vizing [11] that for any simple graph G , $\chi'(G)$ is either $\Delta(G)$ or $\Delta(G) + 1$. The problem of determining the chromatic index of an arbitrary graph is a difficult task. Holyer [6] has proved that edge coloring problem is NP -complete. In this paper, we consider skew edge coloring problems that are inspired from the study of skew Room squares. The concept of skew chromatic index was introduced by Marsha F. Foregger and better upper bounds for $s(G)$ was discussed when G is cyclic, cubic or bipartite [4]. All the notations and definitions used in this paper are as in [2]. A skew edge coloring of a graph G is an assignment of an ordered pair of colors (a_i, b_i) to each edge e_i of G such that the a_i 's form an edge coloring of G , the b_i 's form an edge coloring of G , and the pairs $\{a_i, b_i\}$ are all distinct. The two edge colorings are referred to as component colorings of the skew edge coloring. The skew chromatic index $s(G)$ is the minimum number of colors required for a skew edge coloring of G . For example, $s(K_3) = 3$ where the first and the second component colorings of K_3 are 1, 2, 3 and 2, 3, 1 respectively. It can be observed that not less than three colors can be used for skew edge coloring of K_3 . See Figure 1.

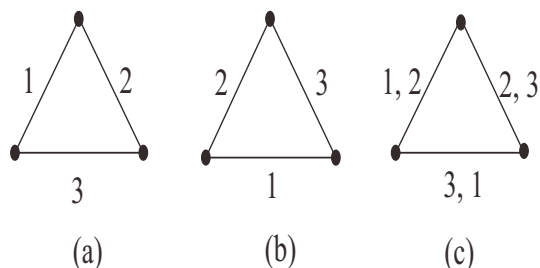


Figure 1: (a) First component coloring of K_3 . (b) Second component coloring of K_3 . (c) Skew edge coloring of K_3 .

In this paper, we develop a common algorithm for skew edge coloring of 2-rooted sibling trees and cyclic snake graphs and the skew chromatic index of these graphs are also solved in polynomial time.

2. Lower Bound on $s(G)$

Skew chromatic index, $s(G)$ is the minimum number of colors used in two edge colorings of G such that no two edges are assigned the same unordered pair of colors. Since each component coloring of a skew edge coloring is itself an edge coloring we have $s(G) \geq \chi'(G)$. But by Vizing's lemma, we have $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.

Hence $s(G) \geq \Delta(G)$. Thus if ‘ k ’ colors are used for skew edge coloring, then there are $\binom{k+1}{2}$ unordered pairs of colors and this number must be at least as large as the number of edges in G . Let $k(m)$ denote the smallest integer ‘ k ’ satisfying $\binom{k+1}{2} \geq m$ where ‘ m ’ denotes the number of edges in G . Thus the best lower bound for $s(G)$ is $s(G) \geq \max\{\Delta(G), k(|E(G)|)\}$ [4]. Our strategy is to obtain $s(G)$ for a given graph G and prove that the lower bound given here is sharp for the graph.

3. Sibling trees and cyclic snake graphs

In this section, we develop a common algorithm for skew edge coloring of 2–rooted sibling trees and cyclic snake graphs. A tree is a connected graph that contains no cycles. The most common type of tree is the binary tree. It is so named because each node can have at most two descendants. A binary tree is said to be a complete binary tree if each internal node has exactly two descendants. These descendants are described as left and right children of the parent node. Binary trees are widely used in data structures because they are easily stored, easily manipulated, and easily retrieved. Also, many operations such as searching and storing can be easily performed on tree data structures. Furthermore, binary trees appear in communication pattern of divide-and-conquer type algorithms, functional and logic programming, and graph algorithms. A rooted tree represents a data structure with a hierarchical relationship among its various elements. Tree based networks have fixed degree nodes and are suitable for massively parallel systems. There are many tree derived architectures like hyper tree, shuffle hyper tree, X –tree, brother tree, slim tree. In this section, we consider one such tree derived architecture called 2–rooted sibling tree and determine its skew chromatic index.

Definition 3.1. [10] *For any non-negative integer r , the complete binary tree of height r , denoted by T_r , is the binary tree where each internal vertex has exactly two children and all the leaves are at the same level. A complete binary tree T_r has r levels and contains 2^{i-1} vertices at level i , $1 \leq i \leq r$.*

Definition 3.2. [10] *The 1–rooted sibling tree ST_r^1 is obtained from the 1–rooted complete binary tree T_r^1 by adding edges (sibling edges) between left and right children of the same parent node. The n –rooted sibling tree ST_r^n is obtained by taking n vertex disjoint 1–rooted sibling tree ST_r^1 on 2^r vertices with roots say $r_1, r_2, \dots, r_n$ and adding edges (r_i, r_{i+1}) , $1 \leq i \leq n-1$. See Figure 2.*

3.2. Cyclic snake graph

A block of a graph G is a maximal connected sub graph of G that has no cut vertex. The block-cut-point-graph of G is a bipartite graph H in which one partite set consists of the cut-vertices of G , and the other has a vertex b_i for each block B_i of G .

Rosa [8] has defined a triangular snake TS_n as a connected graph in which all blocks are triangles and the block-cut-point graph is a path. A triangular cactus is a connected graph all of whose blocks are triangles [1].

Definition 3.3. A triangular snake TS_n with n blocks is a graph obtained from a path $u_1, u_2, \dots, u_n, u_{n+1}$ by joining u_i and u_{i+1} to a new vertex $v_i, i = 1, 2, 3, \dots, n$.

Definition 3.4. A diamond snake graph DS_n [5] is obtained by joining vertices v_i and v_{i+1} to two new vertices u_i and w_i for $i = 1, 2, 3, \dots, n$. It is also known as diamond necklace. A diamond snake graph has $3n + 1$ vertices and $4n$ edges, where n is the number of blocks in the diamond necklace.

Definition 3.5. [7] A bC_n -snake is defined as a connected graph in which all the blocks are isomorphic to the cycle C_n and the block-cut point graph is a path P , where P is the path of minimum length that contains all the cut vertices of a bC_n -snake and b is the number of blocks. It is a graph with $(n - 1)b + 1$ vertices and nb edges.

3.3. Construction of the edge set bC_n .

Construct the edge set $E(G) = \{e_1, e_2, e_3, \dots, e_m\}$ as follows:

Let $V(bC_n) = \{v_{i,j}, 1 \leq i \leq b, 1 \leq j \leq n\}$. It can be noted that $v_{i,1} = v_{i+1,2}, i = 1, 3, 5, \dots, b - 1$, if b is even and $v_{i,1} = v_{i+1,2}, i = 1, 3, 5, \dots, b - 2$, if b is odd.

Also, $v_{i+1,2} = v_{i,3}, i = 2, 4, 6, \dots, b - 2$, if b is even and $v_{i+1,2} = v_{i,3}, i = 2, 4, 6, \dots, b - 1$, if b is odd.

Starting with the leftmost block of the cyclic snake, one block is taken at a time and the edges of every block are ordered in anticlockwise direction. The edge set is given by

$$E(G) = E(bC_n) = \{e_{(i-1)n+j} = v_{i,j}v_{i,j+1}, e_{in} = v_{i,n}v_{i,1}, 1 \leq i \leq b, 1 \leq j \leq n - 1\}.$$

Figure 4 shows the labeling of edges of $8C_{10}$ in order to obtain its skew edge coloring.

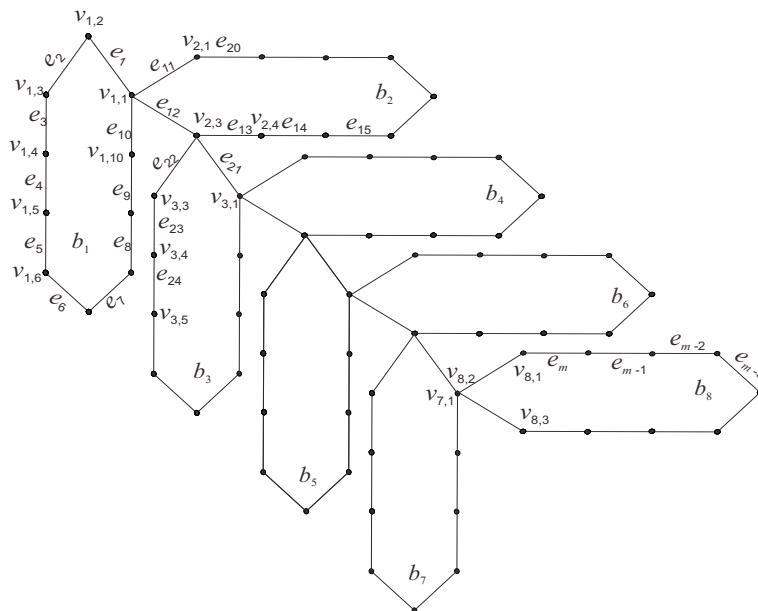


Figure 4: Generalised cyclic snake graph bC_n .

Algorithm for skew edge coloring.

We now present an algorithm for skew edge coloring of 2-rooted sibling tree and cyclic snake graphs.

Input: A graph $G = ST_r^2, r \geq 4$ or a cyclic snake $G = bC_n, n \geq 5$ and $b \geq \left\lfloor \frac{n^2+7n+12}{2n} \right\rfloor$.

Step 1: Find the smallest positive integer ‘ k ’ satisfying $\binom{k+1}{2} \geq m$, where ‘ m ’ denotes the number of edges of the graph G .

Step 2: Consider the set of colors $\{1, 2, 3, \dots, k\}$.

Step 3: Form the set U of distinct unordered pairs of colors defined by $U = \left\{ \{c, c+t\}_{c=1,2,3,\dots,k-t} \cup \{k-t+i, i\}_{i=1,2,3,\dots,t} \right\}_{t=0,1,2,3,\dots, \left\lfloor \frac{k}{2} \right\rfloor}$.

Step 4: Define the set of distinct pairs of colors $S = \{p_1, p_2, \dots, p_m\} \subseteq U$ where p_i ’s are the first m elements of U taken in order.

Step 5: Construct the edge set $E(G) = \{e_1, e_2, e_3, \dots, e_m\}$ as mentioned in section 3.1 or 3.3.

Step 6: Color G using the mapping defined by $f : E(G) \rightarrow S$ such that $f(e_i) = p_i, 1 \leq i \leq m$.

Output: Skew edge coloring of G .

Proof of correctness. Since there are m edges, find the smallest positive inte-

ger ‘ k ’ in such a way that $\binom{k+1}{2} \geq m$, so that $\binom{k+1}{2}$ pairs are available for skew edge coloring of the graph. The coloring of edges is done in a specific order as mentioned in Step 5 of the algorithm. First k edges are assigned the colors in such a way that all ordered pairs of the form (c, c) , $c = 1, 2, 3, \dots, k$ are used. In the second set of k edges, the first $k - 1$ edges are assigned the colors of the form $(c, c + 1)$, $c = 1, 2, 3, \dots, k - 1$. The k^{th} edge is assigned the color $(k, 1)$. In the third set of k edges, the first $k - 2$ edges are assigned colors of the form $(c, c + 2)$, $c = 1, 2, 3, \dots, k - 2$. The $(k - 1)^{th}$ edge and the k^{th} edge are assigned the colors $(k - 1, 1)$ and $(k, 2)$ respectively. This process is continued till all the edges are colored and the above method of edge coloring gives skew edge coloring of the graph. See Figure 5 and 6.

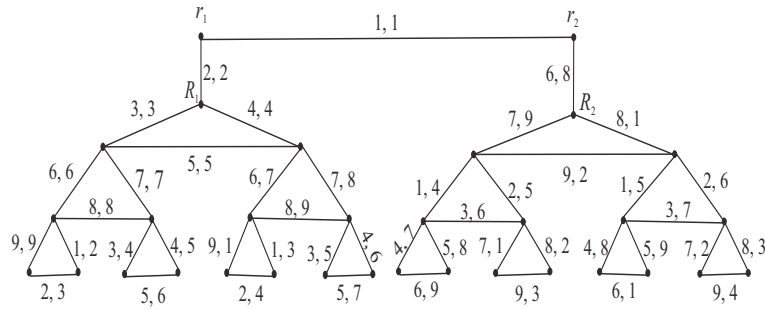


Figure 5: Skew edge coloring of ST_4^2 with $s(G) = 9$.

Theorem 3.6. Let G be a 2-rooted sibling tree ST_r^2 with $r \geq 4$ and $m = 3(2^r - 1)$ edges. Then $s(G) = \left\lceil \frac{-1 + \sqrt{1 + 8m}}{2} \right\rceil$.

Proof. Since there are m edges, at least m pairs of colors are required for skew edge coloring of G . If ‘ k ’ colors are used for coloring, then there are $\binom{k+1}{2}$ unordered pairs. This $\binom{k+1}{2}$ must be at least as large as the number of edges in G . Therefore the smallest positive integer ‘ k ’ is chosen in such a way that $\binom{k+1}{2} \geq m$. i.e., $\frac{(k+1)k}{2} \geq m$. It follows that $k^2 + k \geq 2m$. i.e., $k^2 + k - 2m \geq 0$. Thus solving for ‘ k ’ and taking the positive root, we obtain $k = \frac{-1 + \sqrt{1 + 8m}}{2} \geq 0$, and its greatest integer $k = \left\lceil \frac{-1 + \sqrt{1 + 8m}}{2} \right\rceil$ will be the minimum number of colors used in skew edge coloring.

Therefore $s(G) = k = \left\lceil \frac{-1 + \sqrt{1 + 8m}}{2} \right\rceil = \left\lceil \frac{-1 + \sqrt{1 + 24(2^r - 1)}}{2} \right\rceil$.

Theorem 3.7. Let $G = bC_n$ be a cyclic snake graph with b blocks of C_n , where $n \geq 5, b \geq \left\lceil \frac{n^2 + 7n + 12}{2n} \right\rceil$. Then $s(G) = \left\lceil \frac{-1 + \sqrt{1 + 8bn}}{2} \right\rceil$.

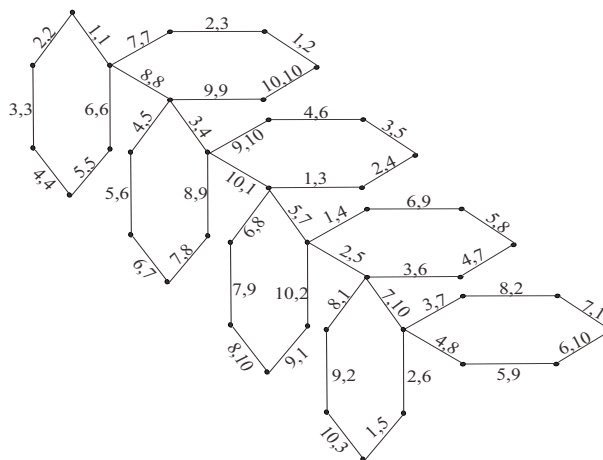


Figure 6: Skew edge coloring of $8C_6$ with $s(G) = 10$.

Thus based on the lower bound for skew edge coloring, we obtain an optimal solution for skew chromatic index of 2–rooted sibling trees and cyclic snake graphs.

4. Time Complexity Analysis of the Algorithm

The pseudo code of the coloring algorithm is given as follows.

Algorithm 3.8. *Skew edge coloring of G .*

Input: Number of edges m , maximum degree $\Delta(G)$ of graph G .

Minimum number of colors $k = \max \left\{ \Delta(G), \left\lceil \frac{-1 + \sqrt{1 + 8m}}{2} \right\rceil \right\}$

count = 1

For $i = 0$ to $k - 1$

For $j = 1$ to k

If count $\leq m$ then

Increment count by 1

If $((j + i) \% k) = 0$ then

Print (j, k)

Else

Print $(j, (j + i) \% k)$

End if

Else

Break

End if

End for

End for

Output: *Skew edge coloring of the graph.*

The time complexity of the algorithm is $O(m)$, where m is the number of edges of the graph. The algorithm runs in linear time as its execution is directly proportional to the input size m .

5. Conclusion

In this paper, an algorithm has been developed for skew edge coloring of 2-rooted sibling trees and cyclic snake graphs and also an optimal solution for $s(G)$ is obtained. It would be interesting to identify the skew chromatic index of other tree derived networks.

References

- [1] Bharati Rajan, Indra Rajasingh, Vasanthi Beulah P., Crossing number of join of triangular snake with mK_1 , path and cycle, *IJCA*, 44(17)(2012).
- [2] Bondy J. A. and Murty U. S. R., *Graph Theory with Applications*, Macmillan, London (1976).
- [3] Fiorini S. and Wilson R. J., *Edge-colorings of graphs*, Pitman, London (1977).
- [4] Foregger M. F., The skew chromatic index of a graph, *Discrete Mathematics*, (49)(1984), 27-39.
- [5] Franklin Thamilselvi M. S., Harmonious coloring of central graphs of certain snake graphs, *Applied Mathematical Science*, 11(12)(2014), 569-578.
- [6] Holyer I., The NP-Completeness of edge-coloring, *Society for Industrial and Applied Mathematics*, (10)(1981), 718-720.
- [7] Lourdusamy A., Sherry George, Super vertex mean labeling of cyclic snakes, *Information in Sciences and Computing*, (2016).
- [8] Rosa A., Cyclic steiner triple systems and labeling of triangular cacti, *Scientia. I*, (1998) 87- 95.
- [9] Sarasvathi V., Iyengar N. CH. S. N., Snehanthu Saha, Interference aware channel assignment using edge coloring in multi-channel multi-radio wireless mesh networks, *Networking and Internet Architecture*, [cs.NI].
- [10] Sundara Rajan, Indra Rajasingh, Rajalaxmi T. M., Parthiban N., Embedding of circulant networks into k -rooted sibling trees, *International Journal of Pure and Applied Mathematics*, 86 (6) (2013), 1005-1012.

- [11] Vizing V. G., On an estimate of the chromatic class of a p -graph (Russian), *Diskret. Analiz.*, (3) (1964), 25-30.